Double Averaged Turbulence Modelling in Eulerian Multi-Phase Flows

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Physics Of Bubbly Flow

Momentum Transfer
- Drag, Lift, Wall Lubrication, and Virtual Mass Forces.
- Turbulent Dispersion Force.
- Responsible for volume fraction distribution.

Turbulence
- Continuous Phase Affects Dispersed Phase
  - Dispersed Phase Reynolds Stresses
  - Turbulent Dispersion
- Presence of Dispersed Phase Affects Continuous Phase
  - Turbulence - Turbulence Modulation
    - Sometimes Enhanced
    - Sometimes Reduced.

Breakup and Coalescence
- Determines Bubble Size Distribution.
  - Breakup caused by, e.g.,
    - turbulence
    - mean shear
    - buoyancy (instabilities)
  - Coalescence caused by, e.g.,
    - high bubble volume fractions
    - increased probability of collisions.
Multi-Phase Turbulence Phenomena

- **Continuous Phase Effects On Dispersed Phase Turbulence.**
  - **Simplest** model for dilute dispersed phase Reynolds stresses:
    \[
    u'_d = C u'_c \\
    \bar{u}'_{id} u'_{jd} = C^2 \bar{u}'_{ic} u'_{jc}
    \]

- **Turbulent Dispersion.**
  - Migration of dispersed phase from regions of high to low void fraction.
  - Continuous phase eddies capture dispersed phase particles by action of interfacial forces.

\[
\nu_{td} = \frac{\nu_{tc}}{\sigma_{v\beta}} \\
\sigma_{v\beta} = \frac{1}{C^2}
\]
Multi-Phase Turbulence Phenomena

- **Dispersed Phase Effects On Continuous Phase Turbulence.**
  - Turbulence Enhancement
    - Due to shear production in wakes behind particles.
  - Turbulence Reduction
    - Due to transfer of turbulence kinetic energy to dispersed phase kinetic energy by action of interfacial forces.
Averaging Procedures

- **First Average = Phase Average**
- Phase indicator function:
  - $\chi_\alpha(x,t) = 1$ if phase $\alpha$ is present, = 0, otherwise.
- Use ensemble-, time- or space-averaging to define phase-averaged variables:
  - ‘Volume Fraction’: $r_\alpha = \langle \chi_\alpha \rangle$
  - Material Density $\rho_\alpha = \langle \chi_\alpha \rho \rangle / r_\alpha$
  - Phase Averaged Transport Variable: $\Phi_\alpha = \langle \chi_\alpha \rho \Phi \rangle / \rho_\alpha$
- Essentially **Mass-Weighted Average**
Averaging Procedures:

- **Phase averaged Momentum and Continuity.**
  \[
  \frac{\partial}{\partial t}(r_\alpha \rho_\alpha U_{\alpha k}) + \frac{\partial}{\partial x_i}(r_\alpha (\rho_\alpha U_{\alpha i} U_{\alpha k} - (\tau_{\alpha ik} + \tau'_{\alpha ik}))) = -r_\alpha \frac{\partial P}{\partial x_k} + r_\alpha B_k + M_{\alpha k}
  \]

- \(M_{\alpha k} = \text{interfacial forces}\)
- \(\tau'_{\alpha ik} = -\rho_\alpha \langle u'_i u'_{\alpha j} \rangle = \text{Reynolds Stress like terms}\)
- Phase induced turbulence, or full turbulence?
- Some researchers assume this represents full turbulence, e.g. Kashiwa et al.
- We assume it represents **phase-induced turbulence**.
Averaging Procedures:

- **Models for Phase-Induced Turbulence.**
- **Sato**: Algebraic Eddy Viscosity:
  \[ \tau_{ij,pi} = -\rho k_{pi} \delta_{ij} + \mu_{t,pi} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial U_i}{\partial x_i} \right) \]
  \[ \mu_{t,pi} = C_{\mu} \rho c r_d d_{p} |U_c - U_d| \]

- Kataoka and Serizawa (1989) derived exact transport equations for \( k_{pi} \) and \( \varepsilon_{pi} \).
- Term identified for enhanced turbulence production:
  \[ P_{k\alpha,pi} = -\bar{M}_{\alpha\beta} \cdot (\bar{U}_\beta - \bar{U}_\alpha) \]
- Exact once the terms for interfacial forces are closed.
Averaging Procedures

- **Second Average = Time or Favre Average.**
- Ensemble averaged phase equations are fully space and time dependent.
- Hence, may apply a second *time- average*.
- **Shear induced turbulence?**
- Favre or Mass Weighted averaging is favoured, as it leads to much fewer terms in the averaged equations.
Favre Averaging

- Favre averaging of phase-averaged variables is defined as follows:
  \[ \tilde{\Phi}_\alpha = \frac{r_\alpha \rho_\alpha \phi_\alpha}{r_\alpha \rho_\alpha} \]

- For constant density phases, reduces to a volume fraction weighted average:
  \[ \tilde{\Phi}_\alpha = \frac{r_\alpha \phi_\alpha}{r_\alpha} \]

- Favre-averaged and time-averaged quantities are related as follows:
  \[ \tilde{\Phi}_\alpha = \Phi_\alpha + \frac{r_\alpha \phi_\alpha'}{r_\alpha} \]
Favre Averaging

- Time- and Favre- averaged velocities are related by:

\[
\tilde{U}_\alpha = \bar{U}_\alpha + \bar{u}_\alpha'' \\
\bar{u}_\alpha'' = \frac{r'_\alpha \bar{u}'_\alpha}{r_\alpha}
\]

- \(r'_\alpha \bar{u}'_\alpha\) is fundamental to turbulent dispersion, as it describes how phasic volume fractions are spread out by velocity fluctuations.

- Eddy-viscosity type turbulence models, employ eddy diffusivity hypothesis (EDH):

\[
r'_\alpha \bar{u}'_\alpha = -\frac{\nu_{i\alpha}}{\sigma_{r\alpha}} \nabla \bar{r}_\alpha
\]

- Turbulent Prandtl number is typically of order unity.
Favre Averaging

- **Time Averaged Continuity Equation**
  \[
  \frac{\partial}{\partial t} \left( \rho_\alpha \overline{r_\alpha} \right) + \frac{\partial}{\partial x_i} \left( \rho_\alpha \left( \overline{U_\alpha r_\alpha} + \overline{r'_\alpha u'_\alpha} \right) \right) = 0
  \]
  - Includes volume fraction-velocity correlation term.
  - Yields additional diffusion term, if we employ the eddy diffusivity hypothesis.

- **Favre Averaged Continuity Equation**
  \[
  \frac{\partial}{\partial t} \left( \rho_\alpha \overline{r_\alpha} \right) + \frac{\partial}{\partial x_i} \left( \rho_\alpha \overline{U_\alpha} \overline{r_\alpha} \right) = 0
  \]
  - No extra terms.
  - A mathematical simplification, not a physical one.
Turbulent Dispersion Force

- Assume caused by interaction between turbulent eddies and inter-phase forces.
- Model using time average of fluctuating part of interphase momentum force.
- Restrict attention to drag force, assumed proportional to slip velocity and interfacial area density $A_{\alpha\beta}$.

$$\vec{M}_\alpha = C_{\alpha\beta} (\vec{U}_\beta - \vec{U}_\alpha) = D_{\alpha\beta} A_{\alpha\beta} \left( \vec{U}_\beta - \vec{U}_\alpha \right)$$

- Assume $D_{\alpha\beta}$ approximately constant as far as averaging procedure is concerned.
Favre Averaged Drag Force

- Express the time averaged drag in terms of Favre averaged velocities.

\[
\overline{M}_\alpha = C_{\alpha\beta} \left( \overline{U}_\beta - \overline{U}_\alpha \right) + \overline{M}_\alpha^{TD}
\]

- Turbulent Dispersion Force (General Form):

\[
\overline{M}_\alpha^{TD} = -\overline{M}_\beta^{TD} = -C_{\alpha\beta} \left( \frac{r_\beta \overline{u}_\beta}{r_\beta} - \frac{r_\alpha \overline{u}_\alpha}{r_\alpha} - \frac{\partial'_{\alpha\beta} \left( \overline{u}'_\beta - \overline{u}'_\alpha \right)}{A_{\alpha\beta}} \right)
\]

- Applicable in this form to flows of arbitrary morphology, using arbitrary turbulence models.

- Modeled Form using EDH:

\[
\overline{M}_\alpha^{TD} = -\overline{M}_\beta^{TD} = C_{\alpha\beta} \left( \frac{v_{i\beta}}{\sigma_{r\beta}} \frac{\nabla r_\beta}{r_\beta} - \frac{v_{i\alpha}}{\sigma_{r\alpha}} \frac{\nabla r_\alpha}{r_\alpha} - \left( \frac{v_{i\beta}}{\sigma_{\Lambda\beta}} - \frac{v_{i\alpha}}{\sigma_{\Lambda\alpha}} \right) \frac{\nabla A_{\alpha\beta}}{A_{\alpha\beta}} \right)
\]
Polydispersed Multi-Phase Flow

- Algebraic form of area density is known:
  \[ A_{\alpha\beta} = \frac{6r_\beta}{d_\beta} \]

- Hence, area density-velocity correlations may be expressed in terms of volume fraction-velocity correlations:

  **General Form:**
  \[
  \vec{M}^{TD}_\alpha = -\vec{M}^{TD}_\beta = C_{\alpha\beta} \left( \frac{r_\alpha u'_\alpha}{r_\alpha} - \frac{r_\beta u'_\alpha}{r_\beta} \right)
  \]

  **Eddy Diffusivity Hypothesis (EDH):**
  \[
  \vec{M}^{TD}_\alpha = -\vec{M}^{TD}_\beta = C_{\alpha\beta} \frac{v_{r\alpha}}{\sigma_{r\alpha}} \left( \frac{\nabla r_\beta}{r_\beta} - \frac{\nabla r_\alpha}{r_\alpha} \right)
  \]
Dispersed Two-Phase Flow

- Further simplifications occur for two phases only:

\[ r_\alpha + r_\beta = 1 \quad \nabla r_\alpha + \nabla r_\beta = 0 \]

- Modeled EDH form of the turbulent dispersion force reduces to a simple volume fraction gradient:

\[
\bar{M}^{TD}_{\alpha} = -\bar{M}^{TD}_{\beta} = -C_{\alpha\beta} \frac{\nu_{i\alpha}}{\sigma_{r\alpha}} \left( \frac{1}{r_\alpha} + \frac{1}{r_\beta} \right) \nabla r_\alpha
\]
Comparison With Other Models

• **Imperial College Model**
  – Behzadi, Issa, and Rusche (ICMF 2001), Effects of turbulence on inter-phase forces in dispersed flow.

• **Chalmers University Model**

• **RPI Models**
  – Lopez de Bertodano, (Ph. D. Thesis, 1992), Turbulent bubbly two-phase flow in a triangular duct., RPI, New York, USA
Imperial College Model

- Idea of modeling turbulence dispersion force by Favre averaging drag term was first proposed by Gosman et al (1992).

- Behzadi et al (ICMF 2001) also consider lift and virtual mass forces, but found them insignificant.

- Equivalent to our model in the dilute limit. \( r_\beta \to 0 \)

- Hence, validation reported by Gosman et al valid for FAD model
Chalmers University Model

\[ \vec{M}_{\alpha}^{TD} = -\vec{M}_{\beta}^{TD} = \beta_1 \nabla r_\beta + \beta_2 \nabla k_\alpha \]

\[ \beta_1 = \frac{\bar{C}_{\alpha\beta} v_{t\beta}}{r_\beta \sigma_{d1}} \]

\[ \beta_2 = \frac{r_\beta \rho_\beta}{\sigma_{d1}} \]

- Similar philosophy and derivation to our model.
- However, requires unconventionally low volume fraction Prandtl numbers, of order 0.001, to achieve reasonable agreement with experiment.
- Due to minor errors in analysis, confusing time-averaging with Favre Averaging.
- Equivalent to our model, if we identify:

\[ \sigma_{d1} = \frac{\sigma_{r\alpha}}{\sigma_{v\beta}} \]

where

\[ v_{t\beta} = \frac{v_{t\alpha}}{\sigma_{v\beta}} \]

- Explains low values of \( \sigma_{d1} \) required to match gas-solid flow.
RPI Models: Lopez de Bertodano

\[ \vec{M}_{\alpha}^{TD} = -\vec{M}_{\beta}^{TD} = -C_{TD} \rho_\alpha k_\alpha \nabla r_\alpha \]

- \( C_{TD} \) is a non-dimensional empirical constant.
  - \( C_{TD} = 0.1 \) to 0.5 gave reasonable results for medium sized bubbles in ellipsoidal particle regime (Lopez de Bertodano et al 1994a, 1994b).

- However, flow regimes involving small bubbles or small solid particles were found to require very different values of \( C_{TD} \), up to 500.

- Revised by Lopez de Bertodano (1999).
  - Proposed that \( C_{TD} \) be expressed as a function of turbulent Stokes number as follows:
  \[ C_{TD} = C_\mu^{1/4} \frac{1}{St(1 + St)} \]
RPI Models: Lopez de Bertodano

- Compare with Favre Averaged Drag (FAD) model for dispersed 2-phase flow employing EVH:

\[
\vec{M}_{\alpha}^{TD} = -\vec{M}_{\beta}^{TD} = -C_{\alpha\beta} \frac{\nu_{i\alpha}}{\sigma_{r\alpha}} \left( \frac{1}{r_{\alpha}} + \frac{1}{r_{\beta}} \right) \nabla r_{\alpha}
\]

- Substitute Eddy Viscosity Formula \( \nu_{i\alpha} = C_{\mu} k_{\alpha}^2 / \varepsilon_{\alpha} \)
- Equivalent to a Lopez de Bertodano model with variable empirical constant:

\[
C_{TD} = \frac{C_{\mu} C_{\alpha\beta}}{\sigma_{r\alpha} \rho_{\alpha} \varepsilon_{\alpha}} \left( \frac{1}{r_{\alpha}} + \frac{1}{r_{\beta}} \right) = \frac{C_{\mu}}{0.41 \sigma_{r\alpha} St} \left( \frac{r_{\beta}}{r_{\alpha}} + 1 \right)
\]

- Strong function of Stokes number, as expected.
RPI Models: Carrica et al

- Requires dispersed phase volume fractions to obey a turbulent diffusion equation in limit where drag + turbulent dispersion balances body forces.

\[
\tilde{M}_{\beta}^{TD} = -\frac{3}{4} C_D \rho_\alpha \left| \bar{U}_\beta - \bar{U}_\alpha \right| \frac{V_{\alpha}}{\sigma_{r\alpha}} \nabla \bar{r}_\beta \quad \beta = 2, \ldots, ND
\]

\[
\bar{M}_{\alpha}^{TD} = -\sum_{\beta=2}^{ND} \tilde{M}_{\beta}^{TD}
\]

- Equivalent to FAD+EDH model in the following limits:
  - Two Phases Only
  - Dilute Dispersed Phase.

- Satisfactory agreement found with DNS data for dilute bubbly flows, and for bubbly mixing layer (Moraga et al, ICMF 2001).
Validation: Bubbly Flow in Vertical Pipe

- Uses Grace Drag Law
- SST turbulence model + Sato eddy viscosity.
- Compares FAD with RPI = constant coefficient Lopez de Bertpdano model.
Liquid-Solid Flow in Mixing Vessel

- Wen Yu drag correlation for dense solids.
- SST turbulence + Sato eddy viscosity.
- Three solid lines are minimum, average and maximum values of CFD results, within region ±5mm from the data point.
- Dimension representative of size of conductivity probe.

(a) \( r = 0.25 \) T; 600 microns

(b) \( r = 0.25 \) T; 710 microns

Jul-04 Dresden, Germany, 2004
Particle volume fractions underpredicted, though correct trends are predicted.

Similar results for 710 micron particles.
Turbulence Modulation

- **Turbulence Enhancement.**
- Due to turbulence production in wakes behind particles.

- Averaged out by first averaging procedure (phase averaging).
- Hence, must include in first averaged equations.
Turbulence Enhancement: Simple Models

- **Sato**: Treat particle-induced and shear-induced turbulence separately.
- **Algebraic Eddy Viscosity model for phase-induced turbulence**:
  \[
  \mu_{t\alpha} = \mu_{t\alpha, si} + \mu_{t\alpha, pi}
  \]
  \[
  \mu_{t\alpha, si} = C_{\mu, si} \rho_{\alpha} \frac{k_{\alpha}^2}{\epsilon_{\alpha}}
  \]
  \[
  \mu_{t, pi} = C_{\mu} \rho c r d_p |U_c - U_d|
  \]

- **Modified k-\(\epsilon\) Models**
  - Lump particle-induced and shear-induced turbulence together.
  \[
  \mu_{t\alpha} = C_{\mu} \rho_{\alpha} \frac{k_{\alpha}^2}{\epsilon_{\alpha}}
  \]
  - Add additional production terms to shear-induced k-\(\epsilon\) equations, e.g. Lee et al.
  \[
  P_{k\alpha} = C_{\alpha\beta} \left( \bar{U}_\beta - \bar{U}_\alpha \right)^2
  \]
  \[
  P_{\epsilon\alpha} = \frac{\epsilon_{\alpha}}{k_{\alpha}} C_1 P_{k\alpha}
  \]
Turbulence Reduction

- Energy transferred from turbulent eddies to particles by acceleration of particles due to drag.
- **Turbulence-drag interaction**, like dispersion.
- Interaction with shear-induced turbulence, so appears as additional source terms in 2nd averaged $k$-equation

\[
S_{k\alpha} = \bar{M}_\alpha \cdot \bar{u}_\alpha'
\]
\[
S_{k\beta} = \bar{M}_\beta \cdot \bar{u}_\beta'
\]
Turbulence Reduction: Simple Model

- **Chen-Wood**: Consider drag only, and treat $C_{\alpha\beta}$ as constant in averaging procedure:

\[
S_{k\alpha} = C_{\alpha\beta}(\overline{U_\beta} - \overline{U_\beta}) \cdot \overline{u'_\alpha} = C_{\alpha\beta}(\overline{u'_\alpha u'_\beta} - \overline{u'_\alpha u'_\alpha}) = C_{\alpha\beta}(k_{\alpha\beta} - 2k_\alpha)
\]

\[
S_{k\beta} = C_{\alpha\beta}(\overline{U_\alpha} - \overline{U_\beta}) \cdot \overline{u'_\beta} = C_{\alpha\beta}(\overline{u'_\alpha u'_\beta} - \overline{u'_\beta u'_\beta}) = C_{\alpha\beta}(k_{\alpha\beta} - 2k_\beta)
\]

\[
k_{\alpha\beta} = \overline{u'_\alpha u'_\beta} = \text{velocity covariance}
\]

- Sum of sources is **negative** $S_{k\alpha} + S_{k\beta} = -C_{\alpha\beta}(\overline{u'_\alpha} - \overline{u'_\beta})^2 \leq 0$

- Hence, can only model turbulence **reduction**.

- Requires model for velocity covariance:

- **Chen-Wood**: $u'_d = Cu'_c$ \[ u'_i u'_j = C_{i,j} u'_i u'_j \]
Turbulence Enhancement: Proposed Double Averaged Approach

- Treat phase-induced and shear-induced separately, as in Sato model.
- Solve separate transport equations for phase-induced and shear-induced turbulence.
- $k-l$ model for phase-induced turbulence: (Lopez de Bertodano et al)
  \[
  \mu_{t\alpha,pi} = C_{\mu} \rho_{\alpha} \sqrt{k_{\alpha,pi} l_{t\alpha,pi}}
  \]

- Choose
  \[
  k_{pi,\infty} = \frac{1}{4} r_{\alpha} \left| \bar{U}_{\beta} - \bar{U}_{\alpha} \right|^2 \\
  l_{t\alpha,pi} \propto d_{\beta}
  \]
- Matches Sato: \[
  \mu_{t,pi} = C_{\mu} \rho_{c} d_{p} \left| U_{c} - U_{d} \right|
  \]
- Choose time-scale $\tau_{k,pi}$ to match Kataoka-Serizawa production:
  \[
  \frac{\bar{p}_{\alpha} k_{pi,\infty}}{\tau_{k,pi}} = C_{\alpha \beta} \left( \bar{U}_{\beta} - \bar{U}_{\alpha} \right)^2
  \]
- Hence, $\tau_{k,pi}$ proportional to particle relaxation time.
- Time Averaged $k_{pi}$ equation introduces additional terms involving turbulence dispersion force.
- Hence, affected by volume fraction gradients.

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Turbulence Reduction: Proposed Double Average Approach

- As for Chen-Wood, but take area density fluctuations into account in averaging procedure.
- Hence, additional source terms in shear-induced $k$-equation:
  
  \[ S_{k\alpha} = D_{\alpha\beta} A_{\alpha\beta} \overline{(U_\beta - U_\beta)} \cdot \overline{u_\alpha'} = C_{\alpha\beta} (k_{\alpha\beta} - 2k_\alpha) \]  
  + additional terms

- Additional terms proportional to  
  \[ \overline{C_{\alpha\beta} (U_\beta - u_\alpha') \frac{\mu_t}{\sigma_r} \nabla r_\alpha} \]

- Hence, also affected by volume fraction gradients.

- Consider better models for velocity covariance, e.g. transport equation.
Conclusions

- Double Averaging Approach yields a natural model for turbulence dispersion, with wide degree of universality.
- Implemented as default model for turbulence dispersion in CFX-5.7 (2004).
- Also produces potentially fruitful approaches to turbulence modulation.
- Topics for further investigation:
  - How is the model affected by taking into account non-linear dependence of drag on slip velocity?
  - How is the model affected by taking into account volume fraction dependence of the drag coefficient?
  - second order closure models.
  - separated flows.